L^2-spectral invariants and convergent sequences of finite graphs

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Abstract. Using the spectral theory of weakly convergent sequences of finite graphs, we prove the uniform existence of the integrated density of states for a large class of infinite graphs.

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Contents

1 Introduction

The goal of the 2006 Oberwolfach-Mini-Workshop "L^2-spectral invariants and the integrated density of states" was to unify the point of views and approaches in certain areas of geometry and mathematical physics. The aim of our paper is to make the connection between those fields even more explicit. Let us start with a very brief introduction to the theory of integrated density of states.

1.1 Laplace operators on infinite graphs and their integrated densities of states

Let G be an infinite connected graph with bounded vertex degrees. We say that G is **amenable** if there exists a sequence of finite connected spanned subgraphs $\{G_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{|\partial G_n|}{|V(G_n)|} = 0\,,$$

where

$$\partial G_n = \{x \in V(G_n) \mid \text{there exists } y \in V(G)\backslash V(G_n) \text{ such that } (x,y) \in E(G_n)\}\,.$$

Such sequences of subgraphs are called **Følner-sequences**. Now let $\Delta_n : L^2(V(G_n)) \to L^2(V(G_n))$ be the Laplacian operator

$$\Delta_n f(x) = \deg(x) f(x) - \sum_{\{y\,|(x,y)\in E(G_n)\}} f(y)\,,$$

where the degree of x is considered in the subgraph G_n. Then Δ_n is a finite dimensional positive self-adjoint operator. Let

$$N_{\Delta_n}(\lambda) := \frac{|\{\text{eigenvalues of } \Delta_n \text{ not larger than } \lambda \text{ (with multiplicities)}\}|}{|V(G_n)|}\,.$$

We call N_{Δ_n} the **normalized spectral distribution function** of Δ_n. We say that the **integrated density of states** exists for the Laplacian of the graph G if there exists a right continuous monotone function σ such that for any Følner-sequence $\{G_n\}_{n=1}^{\infty}$:

$$\lim_{n\to\infty} N_{\Delta_n}(\lambda) = \sigma(\lambda)\,,$$

if λ is a continuity point of σ. We say that the integrated density of state uniformly exists if $\{N_{\Delta_n}\}_{n=1}^{\infty}$ uniformly converge to σ.

Question 1 *For which amenable graphs G does the integrated density of states exist ?*

Let H be the 2-dimensional lattice and K be the 3-dimensional lattice. Construct a new graph G by identifying a vertex of H with a vertex of K. Then the integrated density of states clearly does not exist for the Laplacian of G. This example suggests that one needs some sort of homogeneity in the local geometry of G.

1.2 The periodic case

Let Γ be a countable group and $L^2(\Gamma)$ be the Hilbert-space of the formal sums $\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma$, where $a_\gamma \in \mathbb{C}$ and $\sum_{\gamma\in\Gamma} |a_\gamma|^2 < \infty$. Notice that Γ unitarily acts on $L^2(\Gamma)$ by
$L_\delta(\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma) = \sum_{\gamma\in\Gamma} a_{\delta^{-1}\gamma} \cdot \gamma$. Hence one can represent the complex group algebra $\mathbb{C}\Gamma$ as bounded operators by left convolutions. The weak closure of $\mathbb{C}\Gamma$ in $B(L^2(\Gamma))$ is the group von Neumann algebra $\mathcal{N}\Gamma$. The group von Neumann algebra has a natural trace:

$$\mathrm{Tr}_{\,\Gamma}(A) = \langle A(1), 1\rangle \,,$$

where $1 \in L^2(\Gamma)$ is identified with the unit element of the group. Let $B \in \mathcal{N}\Gamma$ be a self-adjoint element, then by the spectral theorem of von Neumann

$$B = \int_{-\infty}^{\infty} \lambda \, dE_\lambda^B \,,$$

where $E_\lambda^B = \chi_{[-\infty,\lambda]}(B)$. We can associate a *spectral measure* μ_B to our operator B by

$$\mu_B[-\infty, \lambda] = \sigma_B(\lambda) = \mathrm{Tr}_{\,\Gamma} E_\lambda^B \,.$$

Note that the jumps of σ_B are associated to the eigenspaces of B. Now let Γ be a finitely generated amenable group and $\mathrm{Cay}(\Gamma, S)$ be the Cayley-graph of Γ with respect to a symmetric generating set S. Then the Laplacian of G can be regarded as the element

$$\Delta_G = \|S\|\mathbf{1} - \sum_{s\in S} s \in \mathbb{C}\Gamma \,.$$

Hence the obvious candidate for the integrated density of states is the spectral measure σ_{Δ_G}. In fact one has the following result.

Statement 1 *[3] For the Cayley-graph of an amenable group and a Følner-subgraph sequence $\{G_n\}_{n=1}^\infty$ if $B \in \mathbb{C}(\Gamma)$ is a self-adjoint element then $\{N_{B_n}\|_{n=1}^\infty$ uniformly converges to σ_B, where $B_n = p_n B i_n$ and $p_n : L^2(\Gamma) \to L^2(G_n)$ is the natural projection operator, $i_{F_n} : L^2(F_n) \to L^2(\Gamma)$ is the adjoint of p_{F_n} the natural imbedding operator.*

Similar approximation theorem holds for residually finite groups in the weaker sense. Let Γ be a finitely generated residually finite group with finite index normal subgroups

$$\Gamma \rhd N_1 \rhd N_2 \rhd \dots, \cap_{k=1}^{\infty} N_k = \{1\}.$$

Statement 2 *[12] Let $B \in \mathbb{C}\Gamma$ be a self-adjoint element and let $\pi_k(B) = B_k \in \mathbb{C}(\Gamma/N_k)$ be the associated finite dimensional linear operators, where $\pi_k : \Gamma \to \Gamma/N_k$ are the quotient maps. Then the spectral distribution functions N_{B_k} converge at any continuity point of σ_B.*

According to the Strong Approximation Conjecture of Lück the convergence in Statement 2 is always uniform. Note that the conjecture holds for amenable groups [5].

1.3 The aperiodic case

In [11] (see also [10] for a short exposition) Lenz and Stollmann studied graphs constructed by Delone sets in $\mathbb{R}^n$. In these graphs each neighborhood pattern can be seen in a given frequency but they do not have any sort of global symmetries. Instead of the Laplacians they considered finite range pattern-invariant operators. These operators can be viewed as the aperiodic analogs of the elements of the group algebra. They proved the following result.

Statement 3 *Let G be a graph of a Delone-set and $\{G_n\}_{n=1}^{\infty}$ be a Følner-sequence. Also, let A be self-adjoint finite range pattern-invariant operator on G. Then the normalized spectral distributions $\{N_{A_n}\}_{n=1}^{\infty}$ converge uniformly to an integrated density of states σ_A that does not depend on the choice of the Følner-sequence.*

Note that the weak convergence of the spectral distributions was already established by Kellendonk [8] and by Hof [7]. The obstacle what Lenz and Stollmann had to overcome was the possible discontinuity of the integrated density of states due to the existence of finitely supported eigenfunctions. This phenomenon did not occur in the case of lattices. Note however that for certain amenable groups Grigorchuk and Zuk proved the existence of finitely supported eigenfunctions [6] for the Laplacian operators.

1.4 Our results

In Section 2 we study weakly convergent sequences of finite graphs introduced by Benjamini and Schramm [1]. These graph sequences are exactly the ones for which each neighborhood pattern

can be seen at a certain frequency. We introduce two notions: **strong graph convergence** and **antiexpanders**. Strong graph convergence immediately ensures the uniform convergence of the normalized spectral distributions of the Laplacians. The notion of antiexpander seems to be the right notion of amenability in the world of weakly convergent graph sequences. We prove that one can always pick strongly convergent subsequences from weakly convergent antiexpander sequences and conjecture that weakly convergent antiexpander sequences are actually always strongly convergent. In Section 3 we associate von Neumann algebras to weakly convergent graph sequences via finite range pattern-invariant operator sequences. Note that von Neumann algebra was considered in [11] as well, using the action of R^n on the tiling space generated by the Delone-set. In this paper we do not have any group action only the "*statistical symmetry*" given by the existence of the pattern frequency. We construct a limit operator in our von Neumann algebra for finite range pattern-invariant operator sequences. Generalizing Statement 2 we prove that the spectral distribution measure of the limit operator is the integrated density of states. Then in Section 4 we are dealing with amenable graphs and as our main result we answer Question 1 for a rather large class of graphs generalizing Statement 1 and Statement 3.

Main result *(Theorem 2) If G is an amenable graph such that all the Følner- sequences $\{G_n\}_{n=1}^{\infty}$ are weakly convergent antiexpanders, then for any self-adjoint finite range pattern invariant operator A the integrated density of states σ_A uniformly exists.*

2 Graph sequences of bounded vertex degrees

2.1 The weak convergence of graph sequences

First let us recall the notion of weak convergence of finite graphs due to Benjamini and Schramm in a slightly more general form as in [1]. A **rooted** (d, r, X, S)-graph is a finite simple connected graph G

- with a distinguished vertex x (the root),
- such that $\deg(y) \leq d$ for any $y \in V(G)$,
- such that $d_G(x, z) \leq r$ for any $z \in V(G)$, where d_G denotes the usual shortest path distance,
- the vertices of G are colored by the elements of the set X,

- the directed edges of G are colored by the elements of the set S (that is both $\overrightarrow{(x,y)}$ and $\overrightarrow{(x,y)}$, if $(x,y) \in E(G)$).

In general, we shall call finite graphs with vertices colored by X and directed edges colored by S; (X,S)-graphs. Two rooted (d,r,X,S)-graphs G and H are called **rooted isomorphic** if there exists a graph isomorphism between them mapping root to root, preserving both the vertex-colorings and the edge-colorings.
Let $\mathcal{A}(d,r,X,S)$ denote the finite set of rooted isomorphism classes of rooted (d,r,X,S)-graphs. Now let G be an arbitrary finite (X,S)-graph. Then for any $r \geq 1$ we can associate to G a probability distribution on $\mathcal{A}(d,r,X,S)$ by

$$p_G(\alpha) = \frac{|T(G,\alpha)|}{|V(G)|},$$

where $T(G,\alpha)$ denotes the set of vertices $z \in V(G)$ such that the r-neighbourhood of z; $B_r(z)$ represents the class α.

Definition 2.1 *Let* $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ *be a sequence of finite connected* (X,S)*-graphs such that* $V(G_n) \to \infty$ *as* n *tends to* ∞*. Then we say that* $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ *is* **weakly convergent** *if for any* $r \geq 1$ *and* $\alpha \in \mathcal{A}(d,r,X,S)$*,* $\lim_{n\to\infty} p_{G_n}(\alpha)$ *exists.*

Note that if $|X| = 1$, $|S| = 1$ then this is the usual definition of weak convergence for non-colored graphs.

2.2 Strong convergence

Let G be an (X,S)-graph and $y \in V(G)$. Then the **star** of y, $S_G(y)$ is defined as follows;

- $V(S_G(y))$ consists of y and its neighbours.
- $E(S_G(y))$ consists of the edges between y and its neighbours.
- The coloring of $S_G(y)$ is inherited from G (that is $S_G(y)$ is an (X,S)-graph as well).

Definition 2.2 *Let* G, H *be two not necessarily connected graphs on the vertex set* V*. Then the* δ*-distance of them is defined as*

$$\delta(G,H) := \frac{|\{y \in V \mid S_G(y) \ncong S_H(y)\}|}{|V|}.$$

Note that $\cong$ means rooted (X,S)-colored isomorphism not merely graph isomorphism.

Lemma 2.1 *δ defines a metric on the (X, S)-graphs with vertex set V.*

Proof. Clearly, $\delta(G, H) = 0$ if and only if $G = H$. Also, $\delta(G, H) = \delta(H, G)$. Let us check the triangle inequality. Let G, H, J be three graphs on V.

$$\{y \in V \mid S_G(y) \ncong S_J(y)\} \subseteq \{y \in V \mid S_G(y) \ncong S_H(y)\} \cup \{y \in V \mid S_H(y) \ncong S_J(y)\}.$$

That is $\delta(G, J) \leq \delta(G, H) + \delta(H, J)$. ∎

Let $\sigma \in S(V)$ be a permutation of the vertices. Then H^σ denotes the (X, S)-graph on V, where $(\sigma(x), \sigma(y)) \in E(H^\sigma)$ if and only if $(x, y) \in E(H)$. Also, $\sigma(x)$ in H^σ is colored the same way as x is colored in H, respectively $(\sigma(x), \sigma(y)$ is colored the same way as (x, y) is colored in the graph H. Hence we can define

$$\delta_s(G, H) = \inf_{\sigma \in S(V)} \delta(G, H^\sigma).$$

Lemma 2.2 *δ_s defines a metric on the isometry classes of (X, S)-graphs with vertex set V.*

Proof. Clearly, $G \cong H$ if and only if $\delta_s(G, H) = 0$. Also,

$$\delta_s(G, H) = \inf_{\sigma \in S(V)} \delta(G, H^\sigma) = \inf_{\sigma \in S(V)} \delta(G^{\sigma^{-1}}, H) = \delta_s(H, G).$$

Now let G, H, J be three graphs with vertex set V and let $\delta_s(G, H) = \delta(G, H^{\sigma_1})$ and $\delta_s(H, J) = \delta(H, J^{\sigma_2})$. Then $\delta_s(H, J) = \delta(H^{\sigma_1}, J^{\sigma_2 \sigma_1})$, hence

$$\delta_s(G, J) \leq \delta(G, J^{\sigma_2 \sigma_1}) \leq \delta_s(G, H) + \delta_s(H, J). \quad ∎$$

Now we are ready to define the geometric distance of two arbitrary finite connected (X, S)-graphs. Let G and H be (X, S)-graphs. Also, let q, r be two integers such that $q|V(G)| = r|V(H)|$. Denote by qG the union of q disjoint copies of G. Then $\delta_s(qG, rH)$ is well-defined since the graphs qG and rH can be represented on the same vertex set.

Definition 2.3 *The geometric distance of the finite connected (X, S)-graphs G and H is defined as*

$$\delta_\rho(G, H) = \inf_{\{q, r \,|\, q|V(G)| = r|V(H)|\}} \delta_s(qG, rH).$$

Proposition 2.1 *δ_ρ defines a metric on the set of isomorphism classes of finite connected (X, S)-graphs.*

Proof. Clearly, $\delta_\rho(G,H)=\delta_\rho(H,G)$. Now let us check the triangle inequality. Let $\delta_\rho(G,H)\geq \delta_s(qG,rH)+\epsilon$ and $\delta_\rho(H,J)\geq\delta_s(sH,tJ)+\epsilon$. Obviously, $\delta_s(qG,rH)\geq\delta_s(sqG,srH)$. Thus

$$\delta_\rho(G,H)+\delta_\rho(H,J)\geq\delta_s(sqG,srH)+\delta_s(srH,rtJ)+2\epsilon\geq\delta_s(sqG,rtJ)+2\epsilon\geq\delta_\rho(G,J)+2\epsilon\,.$$

Letting $\epsilon\to 0$ we obtain the triangle inequality. The last step is to prove that if $\delta_\rho(G,H)=0$ then $G\cong H$. First suppose that $|G|<|H|$. Let tH and sG be represented on the same vertex set V. We would like to estimate $\delta(tH,(sG)^\sigma)$. Since $|G|<|H|$ at least one edge in each of the t copies of H connects two different components in $(sG)^\sigma$ hence

$$\delta(tH,(sG)^\sigma)\geq\frac{t}{t|V(H)|}=\frac{1}{|V(H)|}\,.$$

This shows immediately that $\delta_\rho(H,G)\geq\frac{1}{|V(H)|}>0$.
Now let us suppose that $|G|=|H|$, but $G\ncong H$. Suppose that tG and tH are represented on the same vertex set V. Again, we estimate $\delta(tH,(tG)^\sigma)$. For each of the t copies of H;

- there exists an edge connecting two components in $(tG)^\sigma$
- or the vertices of the particular copy are exactly the vertices of a copy of G in $(tG)^\sigma$.

Consequently in each of the t copies there exists at least one vertex such that its star in tH is not isomorphic to its star in $(tG)^\sigma$. Hence $\delta(tH,(tG)^\sigma)\geq\frac{1}{|V(H)|}$. That is $\delta_\rho(G,H)>0$. ∎

2.3 Strong convergence implies weak convergence

Let $\mathbf{G}=\{G_n\}_{n=1}^\infty$ be a sequence of finite connected (X,S)-graphs with vertex degree bound d such that $\{G_n\}_{n=1}^\infty$ is a Cauchy-sequence in the δ_ρ-metric. Then we say that $\mathbf{G}$ is a **strongly convergent graph sequence**.

Proposition 2.2 *If* $\mathbf{G}$ *is strongly convergent then* $\mathbf{G}$ *is weakly convergent as well.*

Proof. Let $r>0$ be a natural number, $\epsilon>0$ be a real number. It is enough to prove that there exists $\delta>0$ depending on r,ϵ and on the uniform degree bound d such that if $\delta_\rho(G,H)<\delta$ for the (X,S)- graphs G,H with vertex degree bound d, then $|p_G(\alpha)-p_H(\alpha)|<\epsilon$ for any $\alpha\in\mathcal{A}(d,r,X,S)$. Let t be the maximal possible number of elements in a finite connected graph with diameter $2r$ and vertex degree bound d. Let $\delta=\frac{\epsilon}{3t}$ and suppose that $p_G(\alpha)-p_H(\alpha)>\epsilon$ for some $\alpha\in\mathcal{A}(d,r,X,S)$ and also $\delta_\rho(G,H)<\delta$. Then there exists l and m, $\frac{l}{m}=\frac{|V(H)|}{|V(G)|}$, such that $\delta(lG,(mH)^\sigma)<2\delta$ (We represent lG and mH on the same vertex set). Note that

$p_{lG}(\alpha) = p_G(\alpha), p_{(mH)^\sigma}(\alpha) = p_H(\alpha)$. Let T_1 (resp. T_2) be the set of vertices in lG (resp in $(mH)^\sigma$ having r-neighbourhood isomorphic to α. According to our assumption

$$|T_1| - |T_2| > \epsilon l |V(G)| \,. \tag{1}$$

The number of vertices x in lG such the star of x in lG is not isomorphic to its star in $(mH)^\sigma$ is less than $2\delta l|V(G)|$. Denote this set by W. Let $T' \subseteq T_1$ be the set of vertices z such that $B_r(z)$ contains an element of W. Observe, that if $y \in T_1 \backslash T_1'$ then $y \in T_2$. Indeed if $y \in T_1 \backslash T_1'$ then the r-neighborhood of y as an (X,S)-graph in lG is exactly the r-neighborhood of y in $(mH)^\sigma$. For $w \in W$ let S_w be the set of vertices in lG such that if $x \in S_w$ then $w \in B_r(x)$. Clearly, $|S_w| \leq t$. Therefore $|T_1'| \leq 2\delta t l |V(G)|$. Since $2t\delta < \epsilon$ we are in contradiction with (1). ∎

Remark: The motivation for the definition of our geometric graph distance was the graph distance $\delta_\square$ defined in [2]. In [2] the authors studied the convergent sequences of dense graphs and proved (Theorem 4.1) that a sequence of dense graphs is convergent if and only if they form a Cauchy- sequence in the $\delta_\square$ metric.

2.4 Antiexpanders

Antiexpanders (or in other words hyperfinite graph sequences) were introduced in [4].

Definition 2.4 *Let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a sequence of finite, connected graphs with vertex degree bound d. Then G is an* **antiexpander** *if for any $\epsilon > 0$ there exists $K_\epsilon > 0$ such that for any $n \geq 1$ one can remove $\epsilon|E(G_n)|$ edges in G_n such a way that the maximal number of vertices in a component of the remaining graph G_n' is at most K_ϵ.*

The simplest example for an antiexpander sequence is $\{P_n\}_{n=1}^\infty$, where P_n is a path of length n. The reason we call such sequence antiexpander is that if $\mathbf{H} = \{H_n\}_{n=1}^\infty$ is an expander sequence then for some $\epsilon > 0$ by removing not more than $\epsilon|E(G_n)|$ edges from G_n then at least one of the components of the remaining graph will have size at least $\frac{1}{2}|V(G_n)|$. Thus the notion of antiexpanders is indeed the opposite of the notion of expanders.

Proposition 2.3 *If $\mathbf{G} = \{G_n\}_{n=1}^\infty$ is a strongly convergent sequence of vertex degree bound d then G is an antiexpander sequence.*

Proof. First pick a constant $\epsilon > 0$. Since $\mathbf{G}$ is strongly convergent there exists a number n_ϵ such that $\delta_\rho(G_{n_\epsilon}, G_n) < \frac{\epsilon}{2d}$ if $n \geq n_\epsilon$.

Claim: *If $n \geq n_\epsilon$ then one can remove $\epsilon E(G_n)$ edges from G_n such that in the remaining graph G'_n the maximal number of vertices in a component is at most $|V(G_{n_\epsilon})|$.*

Indeed, let us represent pG_{n_ϵ} and qG_n on the same vertex space such a way that $\delta(pG_{n_\epsilon}, qG_n) < \frac{\epsilon}{d}$. Let us denote by T_n the set of vertices x in qG_n such that the star of x in qG_n is not isomorphic to the star of x in pG_{n_ϵ}. By removing edges incident to the vertices of T_n, all the remaining components shall have edges from pG_{n_ϵ}. Thus the maximal number of vertices in a component shall be at most $|V(G_{n_\epsilon})|$. By our assumption,

$$\frac{|T_n|}{q|V(G_n)|} \leq \frac{\epsilon}{d}.$$

Therefore we removed at most $\epsilon q|V(G_n)|$ edges. By the pigeon hole principle, there exists a component of qG_n from which we removed at most $\epsilon|V(G_n)|$ edges. Thus our claim and hence the proposition itself follows. ∎

2.5 Weakly convergent antiexpanders

In the previous subsections we proved that if $\mathbf{G} = \{G_n\}_{n=1}^\infty$ is a strongly convergent sequence of graphs with vertex degree bound d, then $\mathbf{G}$ is a weakly convergent antiexpander system. The following proposition states that at least a partial converse holds.

Proposition 2.4 *Let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a weakly convergent antiexpander sequence of (X,S)-graphs with vertex degree bound d. Then there exists a subsequence $\{G_{n_k}\}_{k=1}^\infty$ that converges strongly as a sequence of (X,S)-graphs.*

Proof. Clearly, it is enough to prove that for any $\epsilon > 0$ there exists a subsequence $\{G_{n_k}\}_{k=1}^\infty$ such that $\delta_\rho(G_{n_l}, G_{n_m}) \leq \epsilon$ for any pair $l, m \geq 1$. Let $\delta_1 = \frac{\epsilon}{100d}$. For any $n \geq 1$ we remove $\delta_1|E(G_n)|$ edges from G_n such that in the remaining graphs $\{G'_n\}_{n=1}^\infty$ even the largest components have at most K vertices. We call a vertex $z \in V(G_n) = V(G'_n)$ **exceptional** if we removed at least one of the edges incident to z. Clearly, the number of exceptional vertices in G_n is not greater than $\delta_1 d|V(G_n)|$. Let $H_1, H_2, \dots, H_{M_K}$ be the isomorphism classes of connected (X,S)-graphs with vertex degree bound d having at most K vertices. For $1 \leq i \leq M_K$ we denote by c_i^n the number of components in G'_n colored-isomorphic to H_i. Let $r_i^n = \frac{c_i^n}{|V(G_n)|}$ for any $1 \leq i \leq M_K$.

Pick a subsequence $\{G_{n_k}\}_{k=1}^{\infty}$ such that for any $1 \leq i \leq M_K$ and $l, m \geq 1$:

$$|r_i^{n_l} - r_i^{n_m}| \leq \delta_2 = \frac{\epsilon}{100 M_K K} \,. \tag{2}$$

Now pick two numbers $l, m \geq 1$. The proposition shall follow from the following lemma.

Lemma 2.3 $\delta_\rho(G_{n_l}, G_{n_m}) \leq \epsilon$.

Proof. First consider the graph Z_l consisting of $|V(G_{n_m})|$ disjoint copies of G_{n_l} and the graph Z_m consisting of $|V(G_{n_l})|$ disjoint copies of G_{n_m}. We assume that the vertex spaces of Z_l and Z_m are the same. Also, we consider the subgraphs Z'_l (resp. Z'_m) consisting of $|V(G_{n_m})|$ (resp. $|V(G_{n_l})|$ copies of G'_{n_l} (resp. G'_{n_m}). In Z'_l (resp. in Z'_m) we have $|V(G_{n_m})|c_i^{n_l}$ (resp. $|V(G_{n_l})|c_i^{n_m}$) components isomorphic to H_i. For $1 \leq i \leq M_K$ let $q_i = \min\{|V(G_{n_m})|c_i^{n_l}, |V(G_{n_l})|c_i^{n_m}\}$. Choose q_i copies of components of Z'_l (resp. of Z'_m) isomorphic to H_i.

If $z \in V(Z_l)$ (resp. $z \in V(Z_m)$) is *not* in the chosen copies for any $1 \leq i \leq M_K$, then call z (resp. w) a **non-matching** vertex. Now construct a permutation σ on the vertices of Z_m that for each $1 \leq i \leq M_K$ maps the chosen q_i copies of Z'_m onto the q_i copies of Z'_l isomorphically. Define σ arbitrarily on the non-matching vertices.

Claim:

$$\delta(Z_l, Z_m^\sigma) \leq \frac{|A| + |B| + |C| + D|}{|V(Z_l)|} \,, \tag{3}$$

where A (resp. B) is the set of non-exceptional vertices in Z_l (resp. in Z_m) and C (resp. D is the set of non-matching vertices in Z_l (resp. in Z_m).

Indeed, if z is not in $A \cup \sigma(B) \cup C \cup \sigma(D)$ then its star in Z_l is the same as its star in Z_m^σ. Now by our earlier observation

$$\max\{|A|, |B|\} \leq \delta_1 d |V(G_{n_l})||V(G_{n_m})| = \delta_1 d |V(Z_l)| \,.$$

For the number of non-matching vertices we have the estimates

$$|C| \leq \sum_{i=1}^{M_K} (|V(G_{n_m})|c_i^{n_l} - q_i)\, K \quad |D| \leq \sum_{i=1}^{M_K} (|V(G_{n_l})|c_i^{n_m} - q_i)\, K \,.$$

Recall that

$$\left| \frac{c_i^{n_l}}{|V(G_{n_l})|} - \frac{c_i^{n_m}}{|V(G_{n_m})|} \right| \leq \delta_2 \,.$$

That is $||V(G_{n_m})|c_i^{n_l} - |V(G_{n_l})|c_i^{n_m}| \leq \delta_2 |V(Z_l)|$. Hence $|V(G_{n_m})|c_i^{n_l} - q_i \leq \delta_2 |V(Z_l)|$. Therefore,

$$\frac{|C|}{|V(Z_l)|} \leq M_K \delta_2 K \quad \text{and} \quad \frac{|D|}{|V(Z_m)|} \leq M_K \delta_2 K \,.$$

Thus by (3)

$$\delta_\rho(G_{n_l}, G_{n_m}) \leq \delta_s(Z_l, Z_m^\sigma) \leq 2d\delta_1 + 2m\delta_2 K \leq \epsilon \,.$$

Hence our proposition follows. ∎

We finish this subsection with a conjecture.

Conjecture 1 *Let* $\mathbf{G} = \{G_n\}_{n=1}^\infty$ *be a weakly convergent antiexpander sequence of uniformly bounded vertex degrees. Then* $\mathbf{G}$ *is strongly convergent as well.*

For the first sight the conjecture might seem to be a little bit too bold, nevertheless it is not very hard to check that Proposition 2.4 together with the Ornstein-Weiss Quasi-Tiling Lemma implies the conjecture for the Følner-subsets of Cayley-graphs of amenable groups.

2.6 Examples of antiexpanders

Graphs with subexponential growth: Recall that a monotone function $f : \mathbb{N} \to \mathbb{N}$ has **subexponential growth** if for any $\beta > 0$ there exists $r_\beta > 0$ such that $f(r) \leq exp(\beta r)$ if $r \geq r_\beta$. We say that a graph G has growth bounded by f if for any $x \in V(G)$ and $r \geq 1$, $|B_r(x)| \leq f(r)$. A graph sequence $\mathbf{G} = \{G_n\}_{n=1}^\infty$ is of subexponential growth if there exists a monotone function $f : \mathbb{N} \to \mathbb{N}$ of subexponential growth such that G_n has growth bounded by f for any $n \geq 1$. The following proposition is due to Jacob Fox and János Pach.

Proposition 2.5 *If* $\mathbf{G} = \{G_n\}_{n=1}^\infty$ *is a graph sequence of subexponential growth with vertex degree bound* d *then it is an antiexpander sequence.*

Fix a constant $\epsilon > 0$.

Lemma 2.4 *For any function* f *of subexponential growth there exists* $R_f > 0$ *such that if a graph* G *has growth bounded by* f *then for some* $1 \leq r \leq R_f$

$$\frac{|B_{r+1}(x)|}{|B_r(x)|} \leq 1 + \epsilon \,.$$

Proof. If the lemma does not hold, then for each $R > 0$ there exists a graph G of growth bounded by f such that for some $x \in V(G) : |B_R(x)| \geq (1+\epsilon)^{R-1}$ which is in contradiction with the subexponential growth condition. ∎

By the vertex degree condition, it is enough to prove that if G is an arbitrary finite graph of growth bounded by f and of vertex degree bound d then there exists $T \subseteq V(G)$, $|T| \leq \epsilon |V(G)|$

such that the spanned subgraph of $V(G)\backslash T$ consists of components containing at most $f(R_f)$ vertices. We use a simple induction. If $|V(G)| = 1$ the statement trivially holds. Suppose that the statement holds for $1 \le i \le n$ and let $|V(G)| = n+1$. Consider a vertex $x \in V(G)$. Then there exists $1 \le r \le R_f$ such that $\frac{|B_{r+1}(x)\backslash B_r(x)|}{|B_r(x)|} \le \epsilon$. Consider $V(G)$ as the disjoint union

$$V(G) = B_r(x) \cup (B_{r+1}(x)\backslash B_r(x)) \cup Z\,.$$

Obviously the vertices in Z have no adjacent vertex in $B_r(x)$. Let H be the not necessarily connected subgraph spanned by the vertices of Z. By induction, we have a set $S \subset Z$, $|S| \le \epsilon |Z|$ such that the components of the spanned subgraphs of $Z\backslash S$ are containing at most $f(R_f)$ vertices. Now let $T = (B_{r+1}(x)\backslash B_r(x)) \cup S$. The components of the graph spanned by $V(G)\backslash T$ are the components considered above and $B_r(x)$. Clearly, $|B_r(x)| \le f(R_f)$, hence the proposition follows. ∎

Amenability In [4] we proved that if G is a finitely generated residually finite group with a nested sequence of finite index subgroups Γ_n; $\cap_{n=1}^{\infty}\Gamma_n$, then their Cayley-graphs (with respect to generators of Γ) form an antiexpander sequence if and only if Γ is amenable. By the same simple application of the Ornstein-Weiss Quasi-Tiling Lemma [13] one can easily prove that if $\{G_n\}_{n=1}^{\infty}$ is a Folner-sequence in the Cayley graph of a finitely generated amenable group Γ then they form a weakly convergent antiexpander sequence. Note that these graph sequences might have exponential growth.

3 Operators on graph sequences

3.1 The weak convergence of operators

Recall that for a finite graph G the Laplacian $\Delta_G : L^2(V(G)) \to L^2((V(G))$ is a linear operator acting the following way

$$(\Delta_G f)(x) = \deg(x) f(x) - \sum_{x\sim y} f(y)\,.$$

In general, let us consider linear operators A on the vertex set of a finite a graph G given by operator kernels $A : V(G) \times V(G) \to \mathbb{R}$:

$$Af(x) = \sum_{y\in V(G)} A(x,y) f(y)\,.$$

Note that we shall slightly abuse the notation and use the same letter for an operator and its operator kernel. Now let $\alpha \in \mathcal{A}(d,r,X,S)$ and let H be a rooted (X,S)-colored graph

representing the class α. Suppose that $f : V(H) \to \mathbb{R}$ is a function that is invariant under the rooted colored automorphisms of H (i.e. $f(x) = f(\sigma(x))$ if σ is a rooted colored automorphism). Then the function f can be viewed as a function on the vertices of α. We call such a function an **invariant function** on α. Now let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a weakly convergent sequence of (X, S)-graphs of vertex degree bound d and let $\mathbf{A} = A_n : V(G_n) \times V(G_n) \to \mathbb{R}$ be a sequence of operator kernels such that:

- There exists a uniform bound $m_\mathbf{A} > 0$ such that $\sup_{n\in\mathbb{N}} \sup_{x,y\in V(G_n)} |A_n(x, y)| \leq m_\mathbf{A}$.
- There exists a uniform bound $s_\mathbf{A} > 0$ such that for any $n \geq 1$, $x, y \in V(G_n)$, $A_n(x, y) = 0$ if $d_{G_n}(x, y) > s_\mathbf{A}$.
- For the same constant $s_\mathbf{A}$, if $\alpha \in \mathcal{A}(d, s_\mathbf{A}, X, S)$ and $\lim_{n\to\infty} p_{G_n}(\alpha) \neq 0$, then there exists an invariant function $f^\mathbf{A}_\alpha$ on α such that if the $s_\mathbf{A}$-neighborhood of a vertex x represents the class α then $A_n(x, .)$ represents $f^\mathbf{A}_\alpha$ on $B_{s_\mathbf{A}}(x)$. That is the operators depend only on the local patterns.

Then we call the sequence $\mathbf{A} = \{A_n\}_{n=1}^\infty$ be **weakly convergent operator sequence** on $\mathbf{G} = \{G_n\}_{n=1}^\infty$.

Example 1 *Let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a weakly convergent sequence of finite graphs of bounded vertex degrees, then $\{\Delta_{G_n}\}_{n=1}^\infty$ is a weakly convergent sequence of operators.*

Lemma 3.1 *Let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be as above and $\mathbf{A} = \{A_n\}_{n=1}^\infty$ and $\mathbf{B} = \{B_n\}_{n=1}^\infty$ be weakly convergent operator sequences. Then both $\mathbf{AB} = \{A_nB_n\}_{n=1}^\infty$ and $\mathbf{A} + \mathbf{B} = \{A_n + B_n\}_{n=1}^\infty$ are weakly convergent operator sequences. Also, $\mathbf{A}^* = \{A^*_n\}_{n=1}^\infty$ is a weakly convergent operator sequence.*

Proof. Note that

$$A_nB_n(x, y) = \sum_{z\in V(G_n)} A_n(x, z)B_n(z, y)\,. \tag{4}$$

Hence

- $A_nB_n(x, y) = 0$ if $d_{G_n}(x, y) > s_\mathbf{A} + s_\mathbf{B}$.
- $|A_nB_n(x, y)| \leq m_\mathbf{A}m_\mathbf{B}t$, where t is the maximal possible number of vertices of a ball of radius $s_\mathbf{A}$ in a graph G of vertex degree bound d.

Now let $y, z \in V(G_n)$ such that $B_{s_A+s_B}(y) \cong B_{s_A+s_B}(z)$ and these balls are represented by α so that $\lim_{n\to\infty} p_{G_n}(\alpha) \neq 0$. By the pattern invariance assumption and the equation (4), $A_nB_n(y,.)$ and $A_nB_n(z,.)$ represent the same invariant function. This shows that $\{A_nB_n\}_{n=1}^\infty$ is a weakly convergent sequence of operators. The case of $\{A_n+B_n\}_{n=1}^\infty$ can be handled the same way. Now let us turn to the adjoint sequence. Clearly, $A_n^*(x,y) = A_n(y,x)$. Again, suppose that the $2s_{\mathbf{A}}$-neighborhoods of $y, z \in V(G_n)$ are rooted colored isomorphic and represent a class α such that $\lim_{n\to\infty} p_{G_n}(\alpha) \neq 0$. Then it is easy to check that $A^*(y,.)$ and $A^*(z,.)$ represent the same invariant function that is $\{A_n^*\}_{n=1}^\infty$ is a weakly convergent operator sequence. ∎

By our previous lemma if $\mathbf{G} = \{G_n\}_{n=1}^\infty$ is a weakly convergent sequence of (X,S) graphs with vertex degree bound d then the weakly convergent operator sequences form a unital $*$-algebra $\mathcal{P}_{\mathbf{G}}$.

3.2 The trace

Let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a weakly convergent sequence of (X,S)-graphs and $P_{\mathbf{G}}$ be the $*$-algebra of weakly convergent operator sequences. We call $\mathbf{B} = \{B_n\}_{n=1}^\infty$ a nulloperator if for any $\alpha \in \mathcal{A}(d, s_{\mathbf{B}}, X, S)$ the associated invariant function $f_\alpha^{\mathbf{B}}$ is zero. Clearly, the nulloperators are exactly those weakly convergent operator sequences, where

$$\lim_{n\to\infty} \frac{|\{(x,y) \mid B_n(x,y) \neq 0\}|}{|V(G_n)|} = 0\,.$$

It is easy to see that the nulloperators form an ideal $\mathcal{N}_{\mathbf{G}}$ in $\mathcal{P}_{\mathbf{G}}$. For a weakly convergent operator sequence $\mathbf{A} = \{A_n\}_{n=1}^\infty$ we define

$$\mathrm{Tr}_{\,\mathbf{G}}(\mathbf{A}) := \lim_{n\to\infty} \frac{1}{|V(G_n)|} \sum_{x\in V(G_n)} A_n(x,x) = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \mathrm{Tr}\,(A_n)\,.$$

Proposition 3.1 *1. The limit in the definition of $Tr_{\mathbf{G}}(\mathbf{A})$ does exist.*

2. $Tr_{\mathbf{G}}$ is a **trace**, *that is a linear functional satisfying $Tr_{\mathbf{G}}(\mathbf{AB}) = Tr_{\mathbf{G}}(\mathbf{BA})$.*

3. $Tr_{\mathbf{G}}$ is faithful on the quotient space $\mathcal{P}_{\mathbf{G}}/\mathcal{N}_{\mathbf{G}}$ that is $Tr_{\mathbf{G}}(\mathbf{A}^*\mathbf{A}) > 0$ if $\mathbf{A} \notin \mathcal{N}_{\mathbf{G}}$ and $Tr_{\mathbf{G}}(\mathbf{B}) = 0$ if $\mathbf{B} \in \mathcal{N}_{\mathbf{G}}$.

Proof. Let $\alpha \in \mathcal{A}(d, s_{\mathbf{A}}, X, S)$ and $T(G_n, \alpha)$ be the set of vertices in $V(G_n)$ such that $B_{s_{\mathbf{A}}}(x)$ is represented by α.

$$\frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} A_n(x,x) = \frac{1}{|V(G_n)|} \sum_{\alpha \in \mathcal{A}(d,s_{\mathbf{A}},X,S)} \Big(\sum_{x \in T(G_n,\mathbf{A})} A_n(x,x) \Big) =$$

$$= \sum_{\{\alpha \in \mathcal{A}(d,s_{\mathbf{A}},X,S) \,|\, \lim_{n\to\infty} p_{G_n}(\alpha) \neq 0\}} p_{G_n}(\alpha) f^{\mathbf{A}}_{\alpha}(r) + o(1)\,,$$

where $f^{\mathbf{A}}_{\alpha}(r)$ is the value of the invariant function at the root. Thus the limit exists and equals to

$$\sum_{\{\alpha \in \mathcal{A}(d,s_{\mathbf{A}},X,S) \,|\, \lim_{n\to\infty} p_{G_n}(\alpha) \neq 0\}} p_{G_n}(\alpha) f^{\mathbf{A}}_{\alpha}(r)\,. \tag{5}$$

Observe that second statement of the proposition follows immediately from the fact that $\mathrm{Tr}\,(A_n B_n) = \mathrm{Tr}\,(B_n A_n)$.

Let $\mathbf{B} \in \mathcal{N}_{\mathbf{G}}$ then clearly

$$\lim_{n\to\infty} \frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} B_n(x,x) = 0$$

hence $\mathrm{Tr}_{\mathbf{G}}(\mathbf{B}) = 0$. Now let $\mathbf{B} \notin \mathcal{N}_{\mathbf{G}}$. Then

$$\frac{1}{|V(G_n)|} \mathrm{Tr}\,(B_n^* B_n) = \frac{1}{|V(G_n)|} \mathrm{Tr}\,(B_n B_n^*) = \frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} \sum_{y \in V(G_n)} B_n(x,y) B_n^*(x,y) =$$

$$= \frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} \Big(\sum_{y \in V(G_n)} B_n(x,y)^2 \Big)\,.$$

If there exists at least one $\alpha \in \mathcal{A}(d, s_{\mathbf{B}}, X, S)$ such that $\lim_{n\to\infty} p_{G_n}(\alpha) \neq 0$ and the associated invariant function $f^{\mathbf{B}}_{\alpha}$ is non-zero then

$$\lim_{n\to\infty} \frac{1}{|V(G_n)|} \mathrm{Tr}\,(B_n^* B_n) \geq \lim_{n\to\infty} p_{G_n}(\alpha) \sum_{v \in V(\alpha)} |f^{\mathbf{B}}_{\alpha}(v)|^2 > 0\,.$$

3.3 The von Neumann algebra of a graph sequence

The von Neumann algebra of $\mathbf{G}$, $N_{\mathbf{G}}$ is constructed by the GNS-construction the usual way. The algebra $\mathcal{P}_{\mathbf{G}}/\mathcal{N}_{\mathbf{G}}$ is a pre-Hilbert space with inner product

$$\langle [\mathbf{A}], [\mathbf{B}] \rangle = \mathrm{Tr}_{\mathbf{G}}(\mathbf{B}^* \mathbf{A})\,,$$

where $[\mathbf{A}]$ denotes the class of $\mathbf{A}$ in $\mathcal{P}_{\mathbf{G}}/\mathcal{N}_{\mathbf{G}}$. Then $L_{[\mathbf{A}]}[\mathbf{B}] = [\mathbf{AB}]$ defines a representation of $\mathcal{P}_{\mathbf{G}}/\mathcal{N}_{\mathbf{G}}$ on this pre-Hilbert space.

Lemma 3.2 *$L_{[\mathbf{A}]}$ is a bounded operator for any $\mathbf{A} \in \mathcal{P}_{\mathbf{G}}$.*

Proof. Let $\mathbf{A}, \mathbf{B} \in P_G$. Denote by $\| \|$ the pre-Hilbert space norm. Then

$$\|\mathbf{B}\|^2 = \operatorname{Tr}_G(\mathbf{B}^*\mathbf{B}) = \operatorname{Tr}_{\mathbf{G}}(\mathbf{B}\mathbf{B}^*) = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \sum_{x\in V(G_n)} (\sum_{y\in V} |\mathbf{B}(x,y)|^2)\,. \tag{6}$$

$$\|L_{\mathbf{A}}\mathbf{B}\|^2 = \langle \mathbf{AB}, \mathbf{AB}\rangle = \operatorname{Tr}_{\mathbf{G}}((\mathbf{AB})^*\mathbf{AB}) = \operatorname{Tr}_{\mathbf{G}}(\mathbf{B}^*\mathbf{A}^*\mathbf{AB}) = \operatorname{Tr}_{\mathbf{G}}(\mathbf{BB}^*\mathbf{A}^*\mathbf{A}) =$$
$$= \lim_{n\to\infty} \frac{1}{|V(G_n)|} \sum_{x\in V(G_n)} (\sum_{y\in V(G_n)} B_n B_n^*(x,y) A_n^* A_n(y,x)) \leq$$
$$\leq \lim_{n\to\infty} \frac{m_{\mathbf{A}^*\mathbf{A}}}{|V(G_n)|} \sum_{x\in V(G_n)} (\sum_{\{y\in V(G_n)\,|\, d_{G_n}(x,y)\leq s_{\mathbf{BB}^*}\}} |B_n B_n^*(x,y)|)\,.$$

Note that

$$|B_n B_n^*(x,y)| = |\sum_{z\in V(G_n)} B_n(x,z) B_n^*(z,y)| \leq \sum_{z\in V(G_n)} |B_n(x,z)||B_n(y,z)| \leq$$
$$\leq \frac{1}{2} \sum_{z\in V(G_n)} (|B_n(x,z)|^2 + |B_n(y,z)|^2)\,.$$

Let $t_x^n := \sum_{y\in V(G_n)} |B_n(x,y)|^2$. Then by (6)

$$\|\mathbf{B}\|^2 = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \sum_{x\in V(G_n)} t_x^n\,.$$

Hence

$$\|L_{\mathbf{A}}\mathbf{B}\|^2 \leq \lim_{n\to\infty} \frac{m_{\mathbf{A}^*\mathbf{A}}}{|V(G_n)|} \sum_{x\in V(G_n)} (\sum_{\{y\in V(G_n)\,|\, d_{G_n}(x,y)\leq s_{\mathbf{BB}^*}\}} \frac{1}{2}(t_x^n + t_y^n)) \leq$$
$$\leq \lim_{n\to\infty} \frac{m_{\mathbf{A}^*\mathbf{A}} T}{|V(G_n)|} \sum_{x\in V(G_n)} t_x^n\,,$$

where T is the maximal possible number of vertices in an ball of radius $s_{\mathbf{BB}^*}$ in a graph of vertex degree bound d. Hence $\|L_{[\mathbf{A}]}[\mathbf{B}]\|^2 \leq m_{\mathbf{A}^*\mathbf{A}} T \|\mathbf{B}\|^2$. ∎

Thus $\mathcal{P}_{\mathbf{G}}/\mathcal{N}_{\mathbf{G}}$ is represented by bounded operators on the Hilbert-space closure. Now the von Neumann algebra $N_{\mathbf{G}}$ is defined as the weak-closure of $\mathcal{P}_{\mathbf{G}}/\mathcal{N}_{\mathbf{G}}$. Then the trace

$$\operatorname{Tr}_{\mathbf{G}}([\mathbf{A}]) = \langle L_{[\mathbf{A}]}\mathbf{1}, \mathbf{1}\rangle$$

extends to $N_{\mathbf{G}}$ as an ultraweakly continuous, faithful trace. If $\mathbf{A} = \{A_n\}_{n=1}^\infty$ is a weakly convergent operator sequence then we call $[\mathbf{A}] \in N_{\mathbf{G}}$ the **limit operator** of $\mathbf{A}$.

3.4 Representation in the von Neumann algebra and the integrated density of states

In the whole subsection let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a weakly convergent subsequence of finite connected graphs with vertex degree bound d and $\mathbf{B} = \{B_n\}_{n=1}^\infty$ be a weakly convergent sequence of self-adjoint operators.

Lemma 3.3 $[\mathbf{B}] \in N_{\mathbf{G}}$ *is a self-adjoint operator.*

Proof.

$$\langle L_{[B]}[X], [Y] \rangle = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \mathrm{Tr}\,(Y_n^* B_n X_n)$$

$$\langle [X], L_{[B]}[Y] \rangle = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \mathrm{Tr}\,((B_n Y_n)^* X_n) = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \mathrm{Tr}\,(Y_n^* B_n X_n) \quad \blacksquare$$

Lemma 3.4 *There exists some* $K > 0$ *depending on* $\mathbf{B}$ *such that* $\|B_n\| \le K$ *for any* $n \ge 1$.

For the unit vectors $f, g \in L^2(V(G_n))$

$$|\langle B_n(f), g \rangle| = |\sum_{x,y \in V(G_n)} B_n(x,y) f(x) g(y)| \le m_{\mathbf{B}} |\sum_{\{x,y \in V(G_n)\,|\, d_{G_n}(x,y) \le s_{\mathbf{B}}\}} f(x) g(y)| \le$$

$$\le m_{\mathbf{B}}\, |\sum_{\{x,y \in V(G_n), d_{G_n}(x,y) \le s_{\mathbf{B}}\}} f^2(x) + g^2(y)|\,.$$

The number of occurrences of $f^2(x)$ resp. $g^2(y)$ on the right hand side is not greater than $t(d, s_{\mathbf{B}})$ the maximal possible number of vertices in a ball of radius $s_{\mathbf{B}}$ of in a graph of vertex degree bound d. Hence

$$\|B_n\| \le 2 m_{\mathbf{B}} t(d, s_{\mathbf{B}})\,. \quad \blacksquare$$

Since $[\mathbf{B}] \in N_{\mathbf{G}}$ is a self-adjoint element of a von Neumann algebra we have the spectral decomposition:

$$[\mathbf{B}] = \int_0^\infty \lambda \, dE_\lambda^{[\mathbf{B}]}\,,$$

where $E_\lambda^{[\mathbf{B}]} \in N_{\mathbf{G}}$ is the associated spectral projections $\chi_{[0,\lambda]}([\mathbf{B}])$. The **spectral measure** $\mu_{[\mathbf{B}]}$ is defined by

$$\mu_{[\mathbf{B}]}([0,\lambda]) := \mathrm{Tr}_{\,\mathbf{G}}(E_\lambda^{[\mathbf{B}]})\,.$$

Note that the spectral distribution of $\mu_{[\mathbf{B}]}$ is just $N_{[\mathbf{B}]}(\lambda) = \mu_{[\mathbf{B}]}([0,\lambda])$. Let us notice that the spectral distributions N_{B_n} also define probability measures on $\mathbb{R}$ by $\mu_{B_n}[-\infty, \lambda] = N_{B_n}(\lambda)$.

Theorem 1 *(the existence of the integrated density of states) The probability measures μ_{B_n} weakly converge to $\mu_{[\mathbf{B}]}$. Thus for any continuity point λ of $N_{[\mathbf{B}]}$ we have the Pastur-Shubin formulas*

$$\lim_{n\to\infty} N_{B_n}(\lambda) = Tr_{\mathbf{G}}(E_\lambda^{[\mathbf{B}]})\,.$$

Proof. If $P \in \mathbb{R}[x]$ is a polynomial then

$$\int_{-a}^{a} P(\lambda)\, d\mu_{B_n}(\lambda) = \frac{1}{|V(G_n)|}\mathrm{Tr}\,(P(B_n))$$

and

$$\int_{-a}^{a} P(\lambda)\, d\mu_{\mathbf{B}}(\lambda) = \frac{1}{|V(G_n)|}\mathrm{Tr}_{\mathbf{G}}(P(\mathbf{B}))\,,$$

where

$$a := \max\{\|[\mathbf{B}]\|, \sup_{n\geq 1}\|B_n\|\}\,.$$

By definition, $\lim_{n\to\infty}\frac{1}{|V(G_n)|}\mathrm{Tr}\,(B_n^k) = \mathrm{Tr}_{\mathbf{G}}([B]^k)$ thus

$$\lim_{n\to\infty}\int_{-a}^{a} P(\lambda)\, d\mu_{B_n}(\lambda) = \int_{-a}^{a} P(\lambda)\, d\mu_{\mathbf{B}}(\lambda)\,.$$

That is $\{\mu_{B_n}\}_{n=1}^\infty$ weakly converge to $\mu_{\mathbf{B}}$. ∎

3.5 Strong convergence of graphs implies the uniform convergence of the spectral distribution functions

The goal of this subsection is to prove the following proposition.

Proposition 3.2 *Let $\mathbf{G} = \{G_n\}_{n=1}^\infty$ be a strongly convergent graph sequence of vertex degree bound d. Let $\mathbf{B} = \{B_n\}_{n=1}^\infty$ be a weakly convergent sequence of operators on $\mathbf{G} = \{G_n\}_{n=1}^\infty$. Then the spectral distribution functions N_{B_n} uniformly converge.*

Proof. First we need an elementary lemma on small rank perturbations.

Lemma 3.5 *Let $C : \mathbb{R}^n \to \mathbb{R}^n$, $D : \mathbb{R}^n \to \mathbb{R}^n$ be self-adjoint linear transformations such that Rank$(C - D) \leq \epsilon n$. Let N_C resp. N_D be the normalized spectral distribution functions of C resp. of D. Then*

$$\|N_C - N_D\|_\infty \leq \epsilon\,.$$

Proof. Since $N_C(\lambda) = N_{C+r\,Id}(\lambda + r)$ we can suppose that both C and D are positive operators. We call $V \subseteq \mathbb{R}^n$ an M_C-subspace if for any $v \in V$: $\|Cv\| \leq \lambda\|v\|$. Let $d_C(\lambda)$ denote the maximal dimension of a M_C-subspace. Observe that $d_C(\lambda) = nN_C(\lambda)$ that is the number of eigenvalues not greater than λ. Indeed, if W^+_λ is the space spanned by the eigenvectors belonging to eigenvalues larger than λ then $V \cap W^+_\lambda = 0$ for any M_C-subspace V. Therefore $\dim_V \leq n - \dim W^+_\lambda$. On the other hand, $\dim W^-_\lambda = n - \dim W^+_\lambda$, where W^-_λ is the subspace spanned by the eigenvectors belonging to eigenvalues not greater than λ. Since W^-_λ is an M_C-subspace $d_C(\lambda) = nN_C(\lambda)$.

If $\mathrm{Rank}(C - D) \leq \epsilon n$, then $\dim\ker(C - D) \geq n - \epsilon n$. Let $Q_\lambda = W^-_\lambda \cap \ker(C - D)$. Then $\dim Q_\lambda \geq \dim W^-_\lambda - \epsilon n$ and Q_λ is an M_D-subspace of $\mathbb{R}^n$. Therefore $|N_C(\lambda) - N_D(\lambda)| \leq \epsilon$. ∎

Now let us return to the proof of our proposition. Suppose that $\delta_\rho(G_n, G_m) < \frac{\epsilon}{C}$, where C is the maximal possible number of vertices of a ball of radius $s_{\mathbf{A}}$ in a graph of vertex degree bound d. Let q, r be natural numbers such that $\delta_s(qG_n, rG_m) < \frac{\epsilon}{C}$. We define the linear operator $\hat{A}_n$ the following way.

- $\hat{A}_n(\hat{x}, \hat{y}) = A_n(x, y)$ if $\hat{x}$ and $\hat{y}$ are the vertices in a component of qG_n corresponding to the vertices x and y.
- $\hat{A}_n(\hat{z}, \hat{w}) = 0$ if $\hat{z}$ and $\hat{w}$ are not in the same component.

Clearly, the normalized spectral distribution of $\hat{A}_n$ is exactly the same as the one of A_n. We define $\hat{A}_m$ similarly on rG_m. Now let $\hat{A}^\sigma_m(\hat{z}, \hat{w})$ defined as $\hat{A}_m(\sigma(\hat{z}), \sigma(\hat{w}))$.

Lemma 3.6 *Suppose that $\delta(qG_n, (rG_m)^\sigma) < \frac{\epsilon}{C}$, then Rank$(\hat{A}_n - \hat{A}^\sigma_m) < \epsilon|V(qG_n)|$.*

Proof. It is enough to prove that

$$\dim\ker(\hat{A}_n - \hat{A}^\sigma_m) > |V(qG_n)| - \epsilon|V(qG_n)|\,.$$

Let $x \in V(qG_n)$ and v_x be the vector in $L^2(qG_n)$ having the value 1 at x and 0 everywhere else. Clearly, if the $s_{\mathbf{A}}$-neighborhood of x in qG_n is the same as the $s_{\mathbf{A}}$ neighborhood of x in $(rG_m)^\sigma$ then $(\hat{A}_n - \hat{A}^\sigma_m)v_x = 0$. If x is not such a vertex, then x in the $s_{\mathbf{A}}$-neighborhood of an exceptional vertex of qG_n. Thus the number of such vertices is not greater than $C\frac{\epsilon}{C}|V(qG_n)|$. Thus the lemma follows. ∎

By Lemma 3.5, the normalized spectral distributions $\{N_{A_n}\}_{n=1}^\infty$ converge, that proves our proposition. ∎

4 Finite range operators on infinite graphs

4.1 Antiexpander graphs

We say that an amenable (see Introduction) (X, S)-graph has **uniform patch frequency** (UPF) if all of its Følnersubgraph sequences are weakly convergent. We call G an an UPF-antiexpander if :

- G is an amenable graph with the UPF-property.
- All Følner subgraph sequences are antiexpanders.

Example 2 *Let Γ be a finitely generated amenable group and $S = \{s_1, s_2, \dots, s_d, s_1^{-1}, s_2^{-1}, \dots, s_d^{-1}\}$ be a symmetric generating system. Consider the Cayley-graph Cay(Γ, S), where*

- $V(\mathit{Cay}(\Gamma, S)) = \Gamma$.
- $(x, y) \in E(\mathit{Cay}(\Gamma, S))$ *if* $\gamma x = y$ *for some* $\gamma \in S$.
  In this case the edge color of (x, y) *is* γ.

Then Cay$(\Gamma, S))$ is an UPF-antiexpander (see the the end of Subsection 2.6).

Example 3 *The Delone-graphs in [10],[11] are antiexpanders (since they have polynomial growth) and they have the UPF-property.*

4.2 Self-similar graphs

The goal of this subsection is to provide ample amount of examples of UPF-antiexpanders, namely **self-similar graphs** (see e.g. [9] and the references therein for similar constructions). First we fix two positive integers d and k. Let G_1 be a finite connected graph with vertex degree bound d with a distinguished subset $S_1 \subset V_1(G_1)$, which we call the set of connecting vertices. Now we consider the graph $\widetilde{G}_1$, which consists of k disjoint copies of G_1 with following additional properties:

- The graph G_1 is identified with the first copy.
- In each copy the vertices associated to a connecting vertex of G_1 is a connecting vertex of the graph $\widetilde{G}_1$.

The graph G_2 is defined by adding some edges to $\widetilde{G}_1$ such that both endpoints of these new edges are connecting vertices. We must also ensure that the resulting graph still has vertex degree bound d. Finally the subset $S_2 \subset V(G_2)$ is chosen as a subset of the connecting vertices of $\widetilde{G}_1$ such that $S_2 \cap V(G_1) = \emptyset$. That is $G_1 \subset G_2$ is a subgraph and the connecting vertices of G_2 are not in G_1. Inductively, suppose that the finite graphs $G_1 \subset G_2 \subset \ldots \subset G_n$ are already defined and the vertex degrees in G_n are not greater than d. Also suppose that a set $S_n \subset V(G_n)$ is given and $S_n \cap V(G_{n-1}) = \emptyset$. Now the graph $\widetilde{G}_n$ consists of k disjoint copies of G_n and

- The graph G_n is identified with the first copy.
- In each copy the vertices associated to a connecting vertex of G_n is a connecting vertex of the graph $\widetilde{G}_n$.

Again, G_{n+1} is constructed by adding edges to $\widetilde{G}_n$ with endpoints which are connecting vertices, preserving the vertex degree bound condition. Then the set of connecting vertices S_{n+1} is chosen as a subset of the connecting vertices of $\widetilde{G}_n$, such that $S_{n+1} \cap V(G_n) = \emptyset$. The union of the graphs $\{G_n\}_{n=1}^{\infty}$ is connected infinite graph with vertex degrees not greater than d. We call the graph G *self-similar* if $\lim_{n\to\infty} \frac{|S_n|}{|V(G_n)|} = 0$.

Proposition 4.1 *Self-similar graphs are UPF-antiexpanders.*

Proof. First of all note that $\partial G_n \subseteq S_n$, hence self-similar graphs are amenable.

Lemma 4.1 *For any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ and $q_\epsilon \in \mathbb{N}$ such that if $H \subset G$ is a finite spanned subgraph and $\frac{|\partial H|}{|V(H)|} < \delta_\epsilon$ then $\delta_\rho(H, G_{q_\epsilon}) < \epsilon$.*

Proof. By our construction, for any $n \geq 1$, $V(G) = \cup_{i=1}^{\infty} V(G_n^i)$, where

- G_n^i (as a spanned subgraph) is isomorphic to G_n.
- $\frac{|\partial G_n^i|}{|V(G_n^i)|} \leq c_n = \frac{|S_n|}{|V(G_n)|}$.

Now we choose q_ϵ so large that $c_{q_\epsilon} < \frac{\epsilon}{2}$. Then let $\delta_\epsilon = \frac{\epsilon}{10|V(G_{q_\epsilon})|}$. We suppose that $\frac{|\partial H|}{|V(H)|} < \delta_\epsilon$. Consider the set of indices I_H such that $i \in I_H$ if and only if $V(G_{q_\epsilon}^i) \cap V(H) \neq \emptyset$. Also, let $i \in J_H \subseteq I_H$ if and only if $V(G_{q_\epsilon}^i) \subset V(H)$. If $i \in J_H$, then we call $G_{q_\epsilon}^i$ an **inner copy** of G_{q_ϵ} in H. Clearly,

$$|J_H||V(G_{q_\epsilon})| \leq |V(H)| \leq |I_H||V(G_{q_\epsilon}^i)|\,. \tag{7}$$

Note that if $i \in I_H \backslash J_H$ then there exists an element of $x \in \partial H \cap V(G_{q_\epsilon})$. Therefore $|I_H \backslash J_H| \leq |\partial H|$. By our assumption,

$$|I_H \backslash J_H| |V(G_{q_\epsilon})| \leq \delta_\epsilon |V(G_{q_\epsilon})| |V(H)| \,.$$

Hence by (7)

$$|V(H)| - |J_H| |V(G_{q_\epsilon})| \leq \delta_\epsilon |V(G_{q_\epsilon})| |V(H)| \,. \tag{8}$$

Now we estimate the geometric graph distance of H and G_{q_ϵ}. First consider $|V(G_{q_\epsilon})|$ disjoint copies of H and $|V(H)|$ disjoint copies of G_{q_ϵ} on the same vertex set V of cardinality $|V(G_{q_\epsilon})||V(H)|$. Choose a permutation $\sigma \in S(V)$ that maps each of the $|J_H||V(G_{q_\epsilon})|$ inner copies of G_{q_ϵ} in $|V(G_{q_\epsilon})|H$ isomorphically into one of the $|V(H)|$ copies of G_{q_ϵ}. Observe that if the star of a vertex $x \in V$ is not the same in $|V(G_{q_\epsilon})|H$ as in $|V(H)|G_{q_\epsilon}$ then

- either x is not in one of the inner copies
- or x is on the boundary of one of the inner copies.

Hence by (8) and our assumption

$$\delta\left(\left(|V(G_{q_\epsilon})|H\right)^\sigma\right), |V(H)|G_{q_\epsilon}) \leq \frac{\delta_\epsilon |V(G_{q_\epsilon})|^2 |V(H)| + c_{q_\epsilon} |V(G_{q_\epsilon})||V(H)|}{|V(G_{q_\epsilon})||V(H)|} \,.$$

That is

$$\delta_\rho(H, G_{q_\epsilon}) \leq \delta_\epsilon |V(G_{q_\epsilon})| + c_{q_\epsilon} \leq \epsilon \,. \quad \blacksquare$$

Now we turn back to the proof of our proposition. It is enough to prove that if $\{H_n\}_{n=1}^\infty$ is a Følner-subgraph sequence then it is Cauchy in the δ_ρ-metric. However by the previous lemma, if n, m are large enough then $\delta_\rho(H_n, G_{q_\epsilon}) \leq e$ and $\delta_\rho(H_m, G_{q_\epsilon}) \leq \epsilon$, that is $\delta_\rho(H_n, H_m) \leq 2\epsilon$. $\blacksquare$

4.3 The main result

Let G be an infinite connected (X, S)-graph with bounded vertex degrees and $A : V(G) \times V(G) \to \mathbb{R}$ be an operator kernel. We call A a pattern-invariant finite range operator if there exists some s_A such that

- $A(x, y) = 0$ if $d_G(x, y) > s_A$
- $A(x, y) = A(\phi(x), \phi(y))$ if ϕ is a rooted color isomorphism from $B_{s_A}(x)$ to $B_{s_A}(\phi(x))$.

Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ be spanned subgraph sequence in G. Then the **finite volume approximation** of A on $\mathbf{G}$ is the sequence $A_n : V(G_n) \times V(G_n) \to \mathbb{R}$, where $A_n(x, y) = A(x, y)$. In [11] Lenz and Stollmann proved that if A is a pattern-invariant self-adjoint operator on a Delone-graph then the spectral distribution functions of $\{A_n\}_{n=1}^{\infty}$ associated to an arbitrary Følner-sequence converge uniformly to an integrated density of state.
Let $s \in \mathbb{R}\Gamma$ be a self-adjoint element of the group algebra of a finitely generated amenable group Γ, then s defines a pattern-invariant self-adjoint operator A^s on $\mathrm{Cay}(\Gamma, S)$ by

- $A^s(x, y) = c_\gamma$ if $y = \gamma x$ and $s = \sum c_\gamma \gamma$ (note that the self-adjointness of s means that $c_\gamma = c_{\gamma^{-1}}$).

Again we have an associated sequence of finite dimensional approximation for any Folner sequence $\{G_n\}_{n=1}^{\infty}$. Then the spectral distributions of the operators $\{A^s_n\}_{n=1}^{\infty}$ are converge uniformly to the spectral measure of s in the von Neumann algebra of the group Γ [3]. Our main result for infinite graphs generalizes the two results above answering Question 1. for a large family of graphs.

Theorem 2 *Let G be an infinite antiexpander (X, S) graph with uniform patch-frequency. Let A be a self-adjoint finite range pattern-invariant operator on G. Then for any Følner-sequence $\{G_n\}_{n=1}^{\infty}$ the normalized spectral distributions of the finite volume approximations $\mathbf{A} = \{A_n\}_{n=1}^{\infty}$ uniformly converge to an integrated density of state that does not depend on the choice of the Følner-sequence.*

Proof. Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ and $\mathbf{G}' = \{G'_n\}_{n=1}^{\infty}$ be two Følner-sequences in G. with associated approximating operator sequences $\mathbf{A} = \{A_n\}_{n=1}^{\infty}$ and $\mathbf{A}' = \{A'_n\}_{n=1}^{\infty}$. Let $X' = X \cup \{t\}$ and let $H = \{H_n\}_{n=1}^{\infty}$ resp. $H' = \{H'_n\}_{n=1}^{\infty}$ be the (X', S)-graph sequences obtained by recoloring the vertices in ∂G_n resp. in $\partial G'_n$ by the extra color t. Observe that $\mathbf{A}$ resp. $\mathbf{A}'$ are weakly convergent operator sequences on H resp. on H'. Note that if $\alpha \in \mathcal{A}(d, s_A, X', S)$ contains a vertex colored by t then $\lim_{n\to\infty} p_{H_n}(\alpha) = \lim_{n\to\infty} p_{H'_n}(\alpha) = 0$. Also, if $\lim_{n\to\infty} p_{H_n}(\alpha) \neq 0$ then $f^{\mathbf{A}}_\alpha = f^A_\alpha$, where f^A_α is the invariant function on α associated to the finite range operator A.

Lemma 4.2 *The spectral measures of the limit operator $[\mathbf{A}]$ of $\mathbf{A} = \{A_n\}_{n=1}^{\infty}$ in $\mathbb{N}_H$ resp. of the limit operator $[\mathbf{A}']$ of $\mathbf{A}' = \{A'_n\}_{n=1}^{\infty}$ in $\mathbb{N}_{H'}$ coincide.*

By Lemma 5,

$$\mathrm{Tr}_{\mathbf{H}}([\mathbf{A}]^k) = \lim_{n\to\infty} \frac{1}{|V(H_n)|} \mathrm{Tr}\,(A_n^k) = \sum_{\alpha\in\mathcal{A}(d,ks_A,X',S)} \lim_{n\to\infty} p_{H_n}(\alpha) f_\alpha^{\mathbf{A}^k}(r)\,,$$

where $f_\alpha^{\mathbf{A}^k}(r)$ is the value at the root of the invariant function on α associated to $\mathbf{A}^k$. Similarly,

$$\mathrm{Tr}_{\mathbf{H}'}([\mathbf{A}']^k) = \lim_{n\to\infty} \frac{1}{|V(H_n')|} \mathrm{Tr}\,((A_n')^k) = \sum_{\alpha\in\mathcal{A}(d,ks_A,X',S)} \lim_{n\to\infty} p_{H_n}(\alpha) f_\alpha^{\mathbf{A}'^k}(r)\,.$$

By the pattern invariance of the finite range operator A for any $\alpha \in A(d, ks_A, X', S)$,

$$\lim_{n\to\infty} p_{H_n}(\alpha) = \lim_{n\to\infty} p_{H_n'} \quad \text{and} \quad f_\alpha^{\mathbf{A}^k}(r) = f_\alpha^{(\mathbf{A}')^k}(r)\,.$$

Therefore the lemma follows from the definition of the spectral measure. ∎

Consequently, by Theorem 1 we have the following corollary.

Corollary 4.1 *Let G,A, $\mathbf{A} = \{A_n\}_{n=1}^\infty$ be as in Theorem 2. Then the normalized spectral distribution functions $\{N_{A_n}\}_{n=1}^\infty$ converge in any continuity point of a monotone right-continuous function g that does not depend on the choice of the Følner-sequence.*

Now suppose that $\{N_{A_n}\}_{n=1}^\infty$ does not converge uniformly to g. Then there exists a subsequence $\{N_{A_{n_k}}\}_{k=1}^\infty$ such that for each $k \geq 1$,

$$\|N_{A_{n_k}} - g\|_\infty > \epsilon > 0\,.$$

By our previous lemma and Proposition 2.4 we can pick a subsequence of $\{N_{A_{n_k}}\}_{k=1}^\infty$ which uniformly converges. The limit function f is right-continuous and at the continuity points of g, f must coincide with g. Thus $g = f$, leading to a contradiction. This proves Theorem 2. ∎

Remark: If we apply Theorem 2 for Laplacian operators as in Question 1 we encounter a small difficulty. Namely, the finite volume approximation operators of the Laplacian of G on the Følner-subgraphs $\{G_n\}_{n=1}^\infty$ are *not* the Laplacian operators Δ_{G_n} , since the degree of a vertex in the subgraph and in the original graph are different if the vertex is on the boundary. Nevertheless we have the estimate

$$\mathrm{Rank}(p_n\Delta_G i_n - \Delta_{G_n}) \leq |\partial G_n|\,.$$

That is by our Lemma 3.5, the limits of the normalized spectral distributions of $\{p_n\Delta_G i_n\}_{n=1}^\infty$ and of $\{\Delta_{G_n}\}_{n=1}^\infty$ coincide.

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